

Alternating Current Circuits (E6B)

Objectives

- Find the time constant τ of an RL circuit and find the inductance L of a coil.
- Analyze the impedance as a function of frequency.
- Find the resonance frequency of a series RLC circuit.

Theory

A. Faraday's Law

The magnetic flux $\Phi_B = |\vec{B}_n|A = BA \cos \theta$ is defined as the average of the normal component of the magnetic field \vec{B}_n multiplied by the area A , where θ is the angle between \vec{B} and the normal to the surface. You may picture it as the net number of magnetic field lines that are intercepted by the surface A .

The Faraday's law describes the magnitude of the induced *emf*: **“the magnitude of the induced *emf* around a loop is equal to the rate of change of the magnetic flux through the loop”**. Since *emf* is a potential difference, it is measured in Volts.

$$\mathcal{E}(t) = -\frac{\Delta\Phi_B}{\Delta t} = -\frac{\Delta(|\vec{B}_n|A)}{\Delta t} \quad (1)$$

For coils containing N individual turns, the induced voltage is a sum of *emfs* from each individual loop and is given by the following equation.

$$\mathcal{E}(t) = -N\frac{\Delta\Phi_B}{\Delta t} = -N\frac{\Delta(|\vec{B}_n|A)}{\Delta t} \quad (2)$$

The polarity of the induced *emf* can be found using **Lenz's law**: **“the induced current is always such as to oppose the change of flux that generated it”**. The direction of the induced current tends to maintain the original flux through the conducting loop. Lenz's law is indicated by the "-" sign in Faraday's law. It may not be obvious, but the Lenz's law is a consequence of the energy conservation principle.

B. Coil Inductance.

Consider a coil carrying a current I . The current in the coil produces a magnetic field B that varies from point to point, but the value of B at each point is proportional to I . The magnetic flux through the coil is therefore also proportional to current I .

$$\Phi_B = LI \quad (3)$$

where L , the proportionality constant, is called the **inductance** of the coil. The inductance depends on the geometric shape of the coil. For a long coil, the inductance is proportional to the volume of the coil multiplied by the square of the number of turns per unit length. The SI unit of inductance is one **henry** (H).

$$1 \text{ H} = 1 \text{ Wb/A} = 1 \text{ T}\cdot\text{m}^2/\text{A}$$

According to Faraday's law, the induced *emf* for a coil with inductance L can be expressed as

$$\mathcal{E}(t) = -\frac{\Delta\Phi_B}{\Delta t} = -\frac{\Delta(LI)}{\Delta t} = -L \frac{\Delta I}{\Delta t} \quad (4)$$

C. Energy Stored in a Coil.

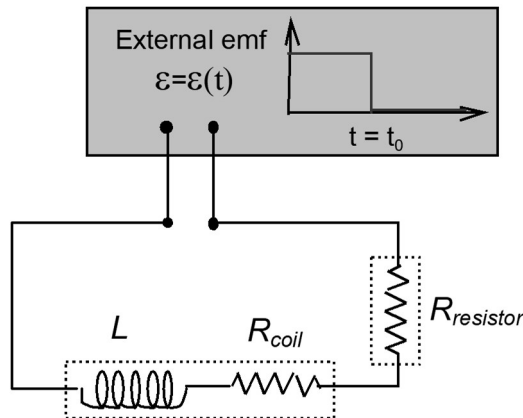
The potential energy stored in a coil with inductance L and current I , is given by the following expression.

$$PE_{ind} = \frac{1}{2} LI^2 \quad (5)$$

The magnetic field B itself stores energy. Therefore, even an empty space without coils, but with magnetic field has a nonzero potential energy. The energy density, which is equal to the energy per unit volume, is described by the next equation.

$$\text{energy density} = \frac{PE_{mag}}{\text{volume}} = \frac{1}{2\mu_0} B^2 \quad (6)$$

D. Current in an RL Circuit.



$$R = R_{resistor} + R_{coil}$$

When an external *emf* is turned off ($\mathcal{E} = 0$), the current in the loop does not disappear instantaneously. Instead, the current decreases gradually from the initial I_{max} value to zero. The time constant τ describes the rate of the current decrease. The larger the inductance L or the smaller the resistance R , the longer it takes the current to reach zero.

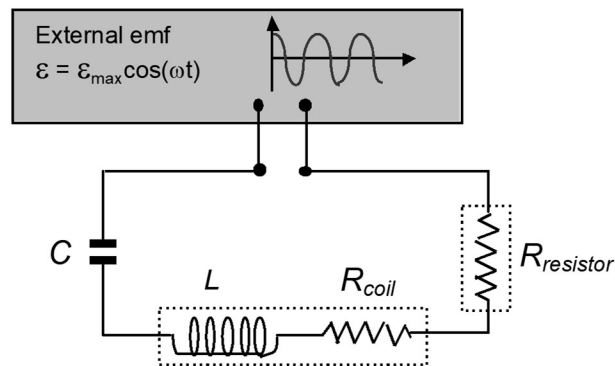
If we assume that the external *emf* was turned off at time $t = t_0$, then the gradual disappearance of current in the RL circuit is given by the exponential function.

$$I(t - t_0) = \varepsilon_{\max}/R * \exp\left(- (t - t_0)/\tau\right) = I_{\max} * \exp\left(- (t - t_0)/\tau\right) \quad (7)$$

where $I_{\max} = \varepsilon_{\text{amplitude}}/R = \varepsilon_{\max}/R$ is the maximum current and $\tau = L/R$ is called the **time constant** for an RL circuit. The time constant is measured in seconds. The inverse of the fitting parameter “*B*” is the time constant τ for the RL circuit.

E. Impedance of an RL Circuit.

Adding capacitor (in series) to an RL circuit creates a series RLC circuit.



In a pure ohmic resistor (R) the voltage waveforms are “in-phase” with the current. In a pure inductance (L) the voltage waveform “leads” the current by $90^\circ = \pi/2$ radians.

For any AC circuit, **impedance** Z is defined as the voltage amplitude divided by the current amplitude. The impedance is a generalized concept of resistance that includes the frequency-dependent behavior of circuits containing inductance (L) and/or capacitance (C).

$$Z = \frac{\varepsilon_{\text{amplitude}}}{I_{\text{amplitude}}} = \frac{\varepsilon_{\max}}{I_{\text{amplitude}}} \quad \text{or} \quad Z = \frac{V_{\text{amplitude of applied voltage}}}{I_{\text{amplitude}}} = \frac{V_{\max}}{I_{\max}} \quad (8)$$

The **impedance** Z contains both the frequency-independent (DC) resistance (R) and the frequency-dependent component called **reactance** (X_L or X_C). For an inductor (e.g., a coil) with inductance L , the reactance is given by the following formula.

$$X_L = 2\pi fL \quad \text{where } f \text{ is the frequency of the current in the AC circuit.} \quad (9)$$

The formula for impedance Z in an RL circuit is shown below.

$$Z = \sqrt{R^2 + X_L^2} = \sqrt{R^2 + (2\pi fL)^2} = \sqrt{(R_{resistor} + R_{coil})^2 + (2\pi fL)^2} \quad (10)$$

where $R = R_{resistor} + R_{coil}$

If $L = 0$, then $Z = \sqrt{R^2 + (2\pi fL)^2} = \sqrt{R^2 + 0} = R$

For direct current (DC), the frequency is equal to zero and the impedance is equal to resistance R .

For DC: $f = 0 \Rightarrow Z = \sqrt{R^2 + 0} = R \quad Z = R$ (only for **direct current** - DC) (11)

For AC: $Z > R$.

F. Impedance of an RC Circuit.

In a pure ohmic resistor (R) the voltage waveforms are “in-phase” with the current. In a pure capacitance (C) the voltage waveform “lags” the current by by $90^\circ = \pi/2$ radians.

For a capacitor C , the reactance is given by the following formula.

$$X_C = \frac{1}{2\pi fC} \quad \text{where } f \text{ is the frequency of the current in the AC circuit.} \quad (12)$$

The formula for impedance in an RC circuit is shown below.

$$Z = \sqrt{R^2 + X_C^2} = \sqrt{R^2 + \left(\frac{1}{2\pi fC}\right)^2} \quad R = R_{resistor} \quad (13)$$

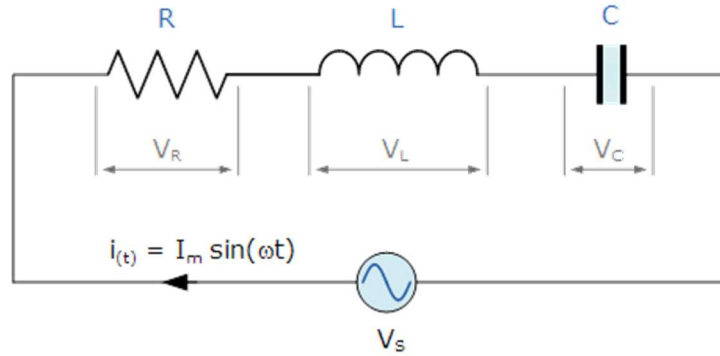
If $f = 0$, then $Z = \infty$, i.e., a charged capacitor does not conduct DC currents.

For large frequencies, i.e., when $f \gg \frac{1}{2\pi C}$, the reactance X_C is very small and the impedance Z is equal to resistance R .

$$\text{If } f \gg \frac{1}{2\pi C} \Rightarrow \frac{1}{2\pi fC} \rightarrow 0 \quad \text{and } Z = R$$

G. Resonance in RLC Circuit.

A single-loop RLC circuits contain two energy storage elements, an inductance L and a capacitance C . Consider the RLC circuit shown below.



The single-loop RLC circuit above has a single loop with the instantaneous current flowing through the loop being the same for each circuit element. Since the inductive and capacitive reactances X_L and X_C are a function of the supply frequency, the sinusoidal response of a series RLC circuit will therefore vary with frequency f . Then the individual voltage drops across each circuit element of R, L and C element will be “out-of-phase” with each other.

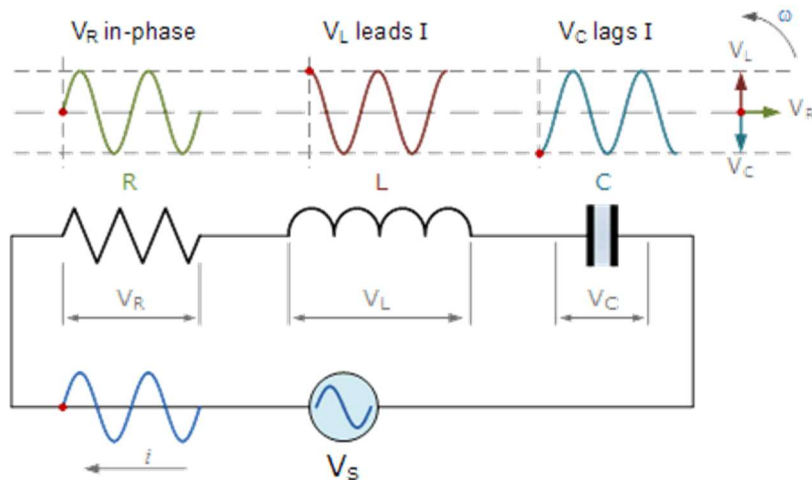
The instantaneous voltage across a pure resistor, V_R is “in-phase” with current $I(t) = I_{max} \sin(\omega t) = I_{max} \sin(2\pi ft)$.

The instantaneous voltage across a pure inductor, V_L “leads” the current by 90° .

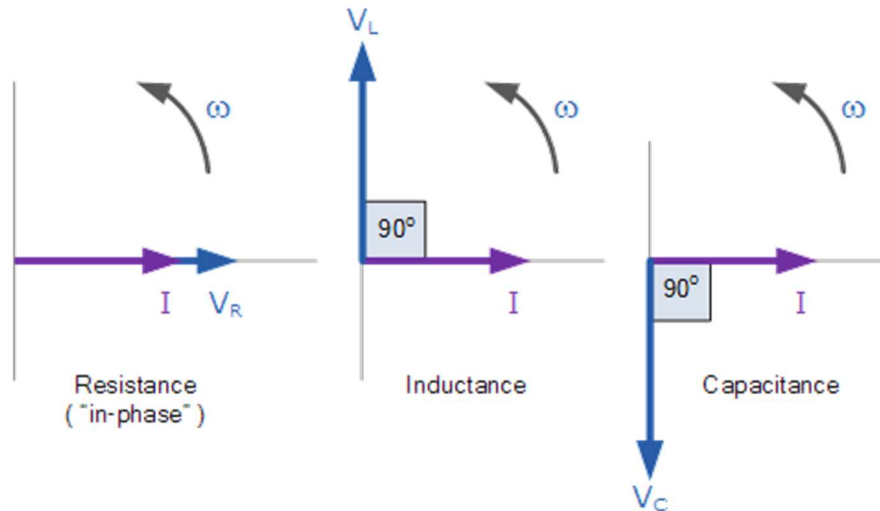
The instantaneous voltage across a pure capacitor, V_C “lags” the current by 90° .

Therefore, V_L and V_C are 180° “out-of-phase” and in opposition to each other.

For the single-loop RLC circuit above, this can be shown as:



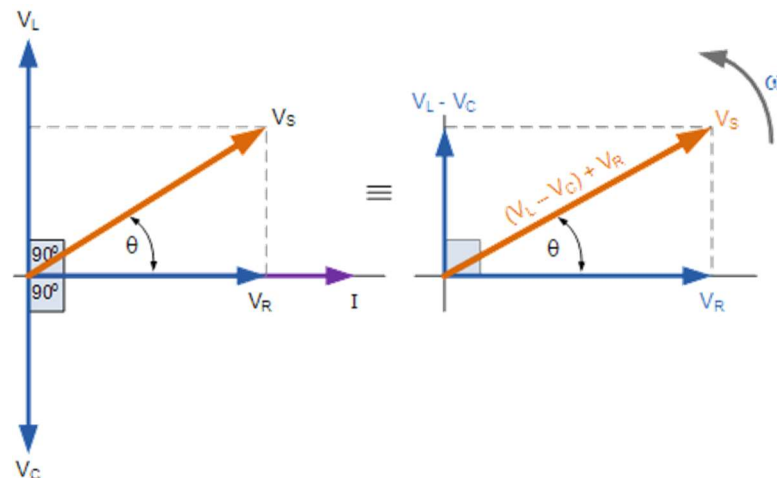
The amplitude of the source voltage across all three components in a series RLC circuit is made up of the three individual component voltages, V_R , V_L and V_C with the current common to all three components. The vector diagrams will therefore have the current vector as their reference with the three voltage vectors being plotted with respect to this reference as shown below.



This means then that we cannot simply add together V_R , V_L and V_C to find the supply voltage, V_S across all three components as all three voltage vectors point in different directions with regards to the current vector. Therefore, we will have to find the supply voltage, V_S as the **phasor sum** of the three component voltages combined together as vectors, i.e., following rules for adding vectors.

Combining together the three individual phasors above and adding these voltages represented as vectors produces the phasor diagram for a series RLC circuit. Since the current flowing through the circuit is common to all three circuit elements, we can use this as the reference vector with the three voltage vectors drawn relative to this at their corresponding angles.

The resulting vector V_S is obtained by adding together two of the vectors, V_L and V_C and then adding this sum to the remaining vector V_R . The resulting angle obtained between V_S and current I will be the circuit's phase angle as shown below.



We can see from the phasor diagram on the right-hand side above that the voltage vectors produce a rectangular triangle, comprising of hypotenuse V_S , horizontal axis V_R and vertical axis $V_L - V_C$. We can obtain the value of V_S as shown.

$$V_S^2 = V_R^2 + (V_L - V_C)^2$$

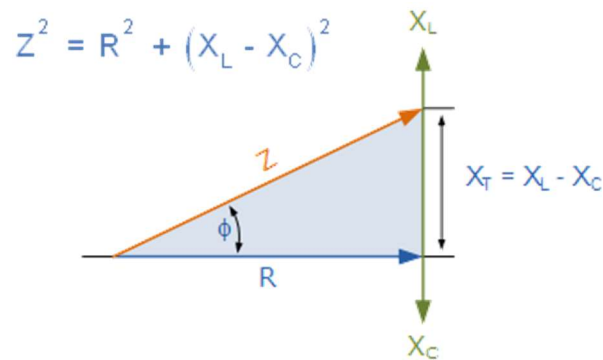
$$V_S = \sqrt{V_R^2 + (V_L - V_C)^2}$$

$$V_S = \sqrt{(IR)^2 + (IX_L - IX_C)^2} = I\sqrt{(R)^2 + (X_L - X_C)^2}$$

This gives us the formula for the impedance Z in a series RLC circuit.

$$Z = V_S/I = \sqrt{(R)^2 + (X_L - X_C)^2} \quad (14)$$

$$Z^2 = (R)^2 + (X_L - X_C)^2 \quad (15)$$



Since both coil reactance $X_L = 2\pi fL$ and capacitor reactance $X_C = 1/(2\pi fC)$ depend on frequency, the total impedance in a single-loop RLC circuit also depend in the frequency f .

$$Z = \sqrt{R^2 + (X_L - X_C)^2} = \sqrt{R^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} = \sqrt{(R_{resistor} + R_{coil})^2 + \left(2\pi fL - \frac{1}{2\pi fC}\right)^2} \quad (16)$$

$$\text{where } R = R_{resistor} + R_{coil}$$

The amplitude of the current in a circuit containing inductance L and capacitance C gets the largest value when the impedance Z has its minimum. Since $R = R_{resistor} + R_{coil}$ does not change with frequency, the minimum of impedance Z occurs for the following frequency f_0 .

$$X_L = X_C \Rightarrow 2\pi fL = \frac{1}{2\pi fC} \quad \text{and} \quad Z = \sqrt{R^2} = R \quad \text{or}$$

$$\left(2\pi fL - \frac{1}{2\pi fC}\right) = 0 \Rightarrow 2\pi fL = \frac{1}{2\pi fC} \Rightarrow f = f_0 = \frac{1}{2\pi\sqrt{LC}} \quad (17)$$

The frequency f_0 corresponding to the maximum current amplitude (the minimum impedance Z) is called the **resonance frequency**.

Procedure:

Activity 1: Time Constant for RL Circuits

Download file files for experiment “(E6B) AC CIRCUITS” and run “E6B, Activity 1”.

- 1.1 First, you need to measure the resistance R_{coil} of the coil attached to the “RLC Circuit” board. Connect the two wires from “OUTPUT” of the data interface box as shown below to terminals **B** and **C** on the “RLC Circuit” board.

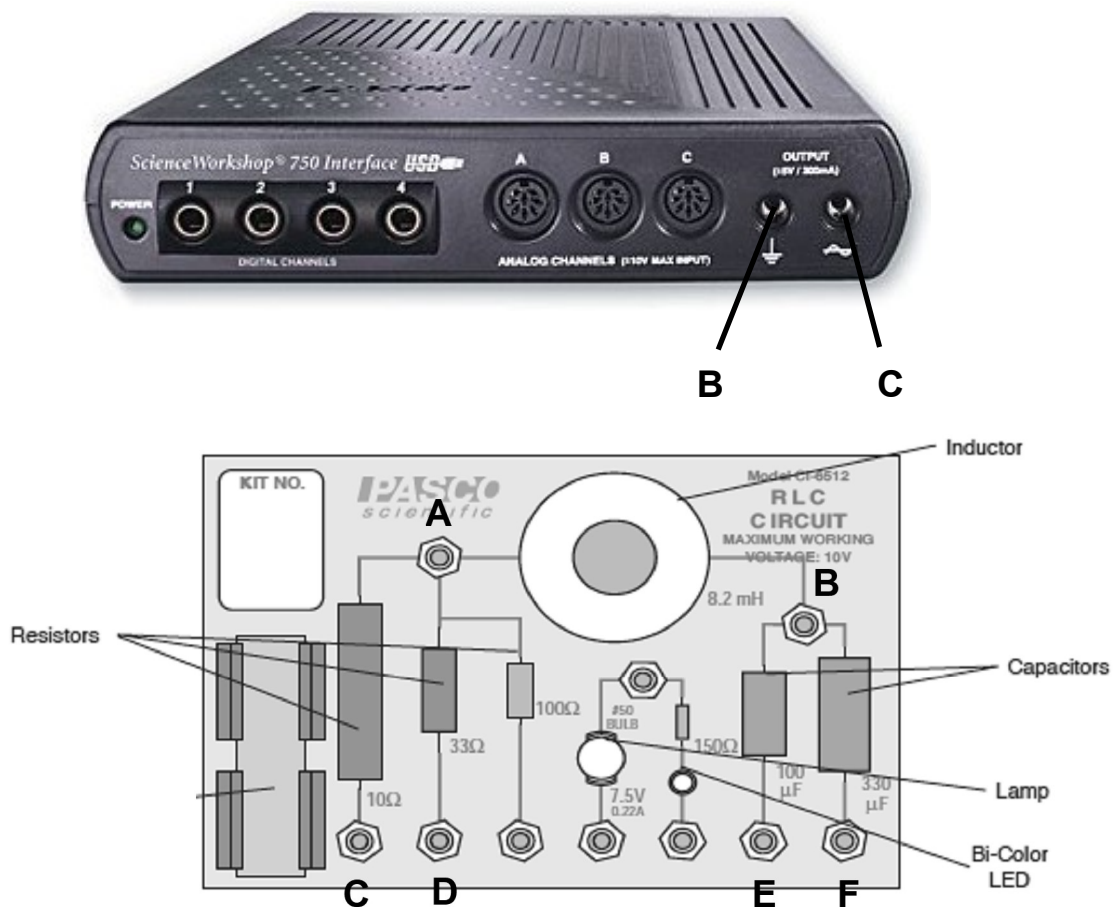
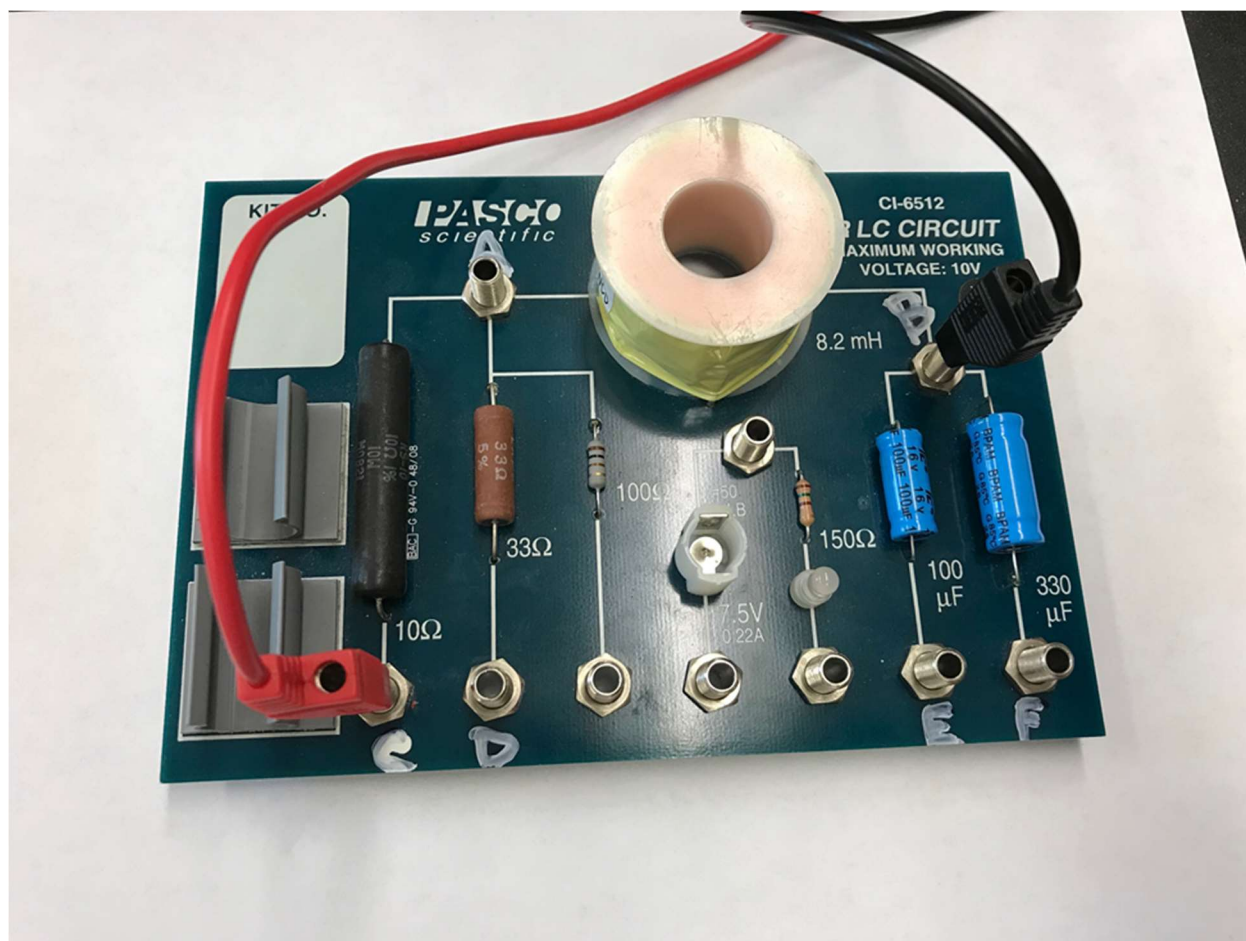



Fig. 1. The “RLC Circuit” Board.

The RL circuit includes now:


- $R_{resistor} = 10.0 \Omega$;
- a coil with unknown inductance L and resistance R_{coil} ;
- no capacitors.

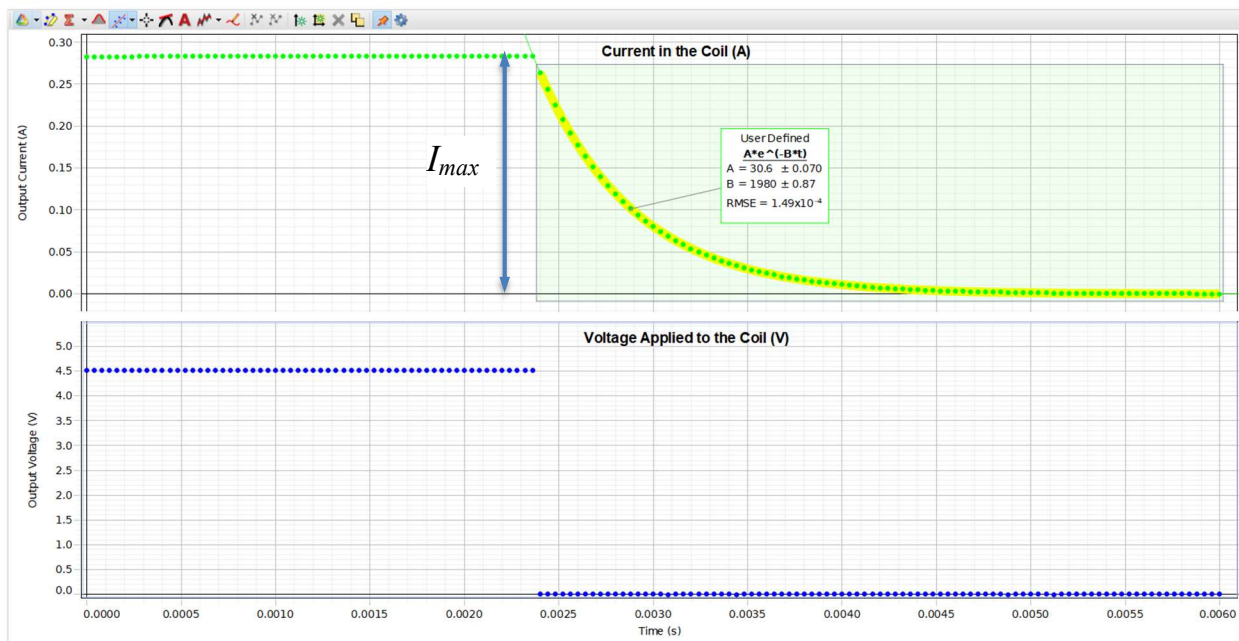


- 1.2. You should see two graphs on the computer monitor. The upper graph shows current versus time and the bottom graph shows the applied voltage from the “OUTPUT” of the data interface box. Click on the **Record** button at the bottom of the screen. The data interface is going to turn on the 4.50 V constant voltage and then turn it off. When the applied voltage is equal to 4.50 V, the current also has a nonzero value. When the applied voltage drops to zero, the current in the circuit does not go to zero instantaneously. Instead, the current decreases exponentially. The rate of decrease is described by the **time constant** τ for the RL circuit (see the Theory section).

Read the exact value of the DC voltage (should be close to 4.50 V) and the maximum value of the current. Click on the “Coordinates” tool  and add “coordinates/delta” tool to read the maximum current value from the graph. Then, use Ohm's law to calculate the resistance (R) of the coil and the 10.0 Ω resistor connected in series. After subtracting the resistor value - 10.0 Ω , you get the resistance of the coil alone (R_{coil}).

- 1.3. Capstone is configured (for this Activity) to find the exponential fit for current vs. time graph and therefore to find the time constant. If no data points are selected on the graph, then computer tries to fit the exponential function to all data points, which is usually wrong.

To select points that computer needs for the fit click anywhere on the “Current in the Coil” graph to make it active and use the “Highlight range of points” tool . Once you drag it into the graph, you should see a rectangle for highlighted points. Move and resize the highlights rectangle to include all data points during the exponential decrease, but eliminate other points when current is constant. The computer should update the fit limiting it to these points. The applied voltage on the lower graph should be equal to zero for these points. Please note that the whole cycle shown on the graph takes only a few milliseconds.



- 1.4. The following used defined function is used for the fit to the selected data points:

$$I(t) = A * \exp(-Bt)$$

This has the same mathematical form as the current vs. time:

$$I(t) = I_{max} * \exp(-t/\tau)$$

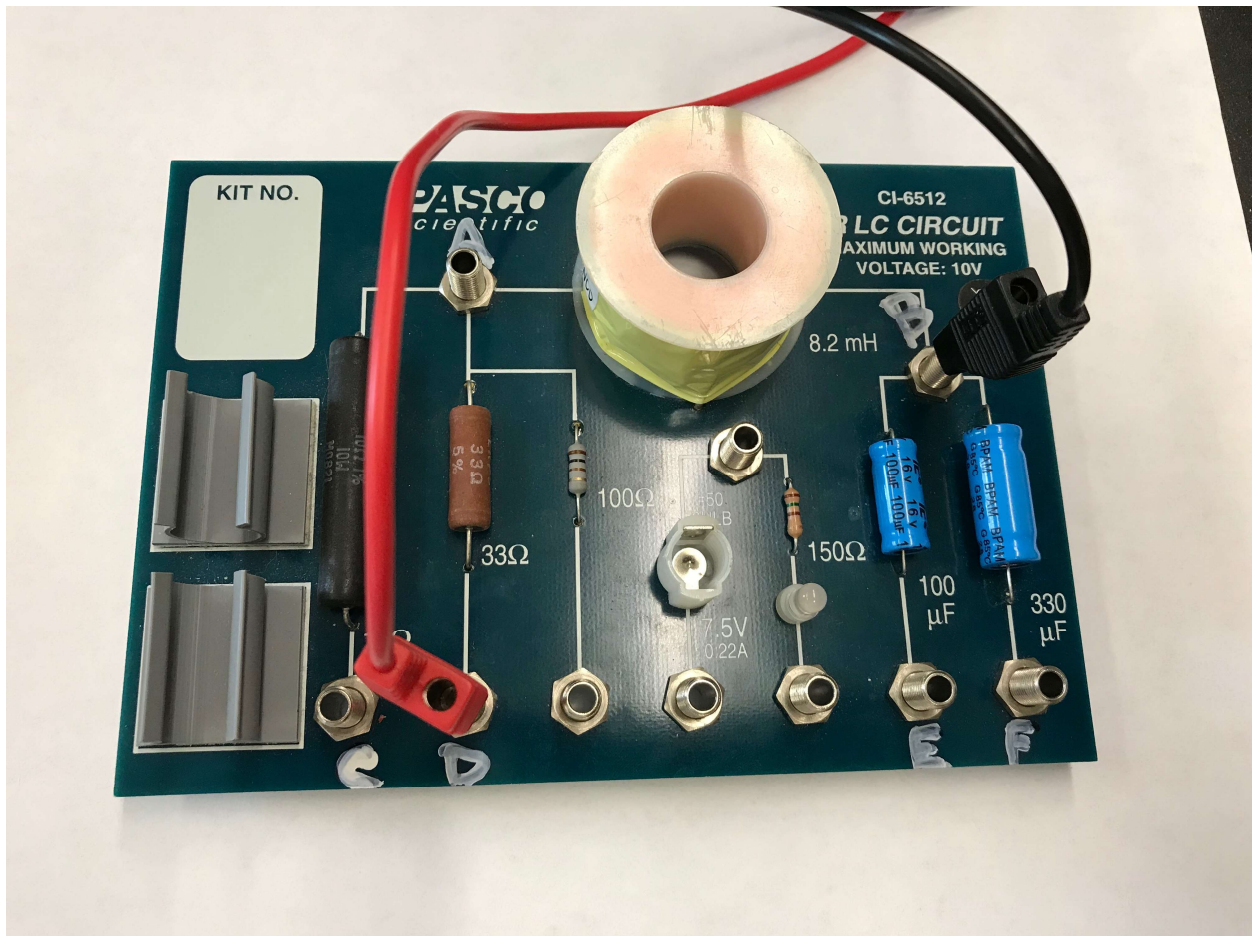
Therefore, the fitting parameter B is equal to $1/(\text{time constant})$.

$$B = 1/\tau \Rightarrow \tau = 1/B,$$

Record the value of the fitting parameter B and the time constant τ for the RL circuit.

Print the graph for $10.0\ \Omega$ resistor and label it “Time constant #1”.

- 1.5. Using the time constant τ and the resistance R in the RL circuit, calculate the inductance L_I of the coil attached to the RLC board.
- 1.6. Calculate the potential energy PE_{ind} stored in the coil's magnetic field when the current is running through the coil and $10.0\ \Omega$ resistor (before the external voltage is turned off). Use equation (5) and the average value of the coil inductance L_{AV} . The current I_{max} was measured in step 1.2.
- 1.7. Next, you should replace the $10.0\ \Omega$ resistor with $33.0\ \Omega$ resistor ($R_{resistor} = 33.0\ \Omega$) in the RL circuit. To do that, move the wire from terminal C to terminal D on the board. Keep the wire connected to terminal B. Calculate the new value of the resistance R in the RL circuit. $R = R_{resistor} + R_{coil}$



- 1.8. Repeat measurements listed in steps 1.3 and 1.4. Do not print the graph for $33.0\ \Omega$ resistor.

Using the time constant τ and the resistance R in the RL circuit, calculate the second value L_2 of the inductance of the coil attached to the RLC board. Do not print this graph. Should this value of inductance be similar to the inductance measured using the RL circuit with 10.0 Ω resistor?

- 1.9. Calculate the average value of the coil inductance L_{AV} .
- 1.10. Calculate the potential energy PE_{ind} stored in the coil's magnetic field when the current is running through the coil and 33.0 Ω resistor (before the external voltage is turned off). Use equation (5) and the average value of the coil inductance L_{AV} . The current I_{max} was measured in step 1.2.

Activity 2: Impedance as a Function of Frequency for RL Circuits

In this activity, you will measure the impedance of an RL circuit as a function of frequency. Later you will compare your results with the theoretical value of the impedance calculated using the average inductance L_{AV} and the resistance R in the RL circuit.

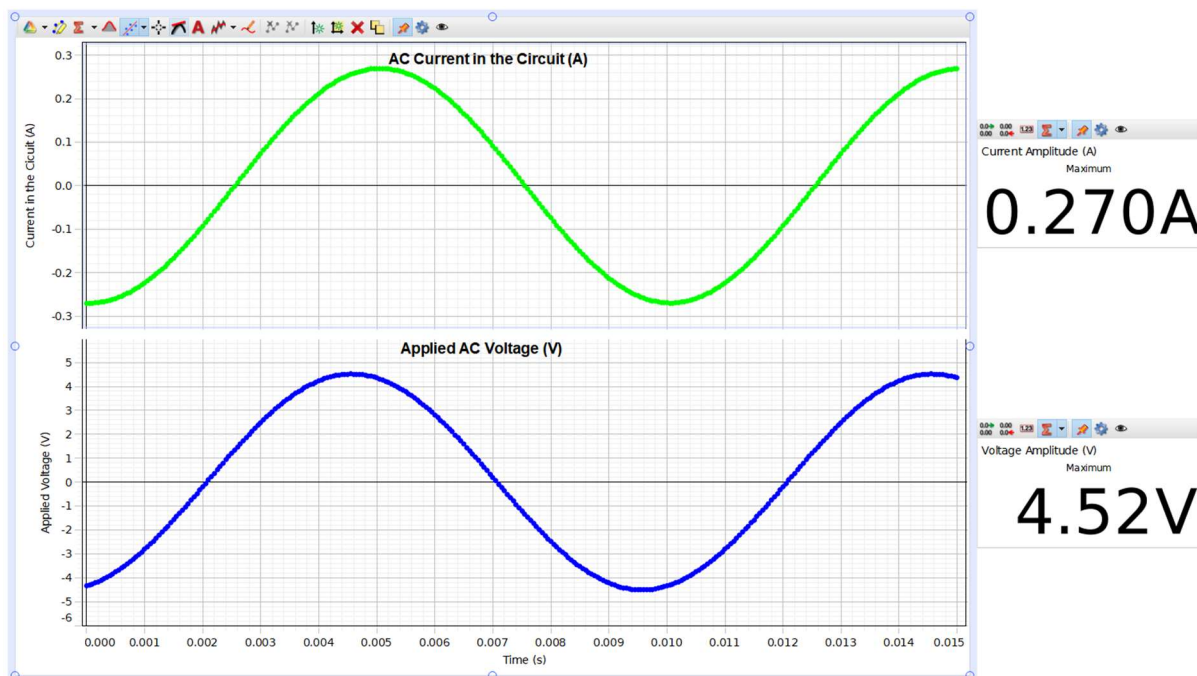
- 2.1. We are going to use 10.0 Ω resistor again, so move the wire from terminal D back to terminal C on the board. Keep the wire connected to terminal **B**.

The RL circuit includes now:

- $R_{resistor} = 10.0 \Omega$;
- a coil with inductance L_{AV} and resistance R_{coil} ;
- no capacitors.

Calculate the resistance R in the RL circuit. $R = R_{resistor} + R_{coil}$

- 2.2 Load the software for activity 2: “E6B, Activity 2”. This software will apply the alternating voltage ($V_{amplitude} = 4.5 \text{ V}$) to the RL circuit. The initial frequency is equal to 100 Hz.

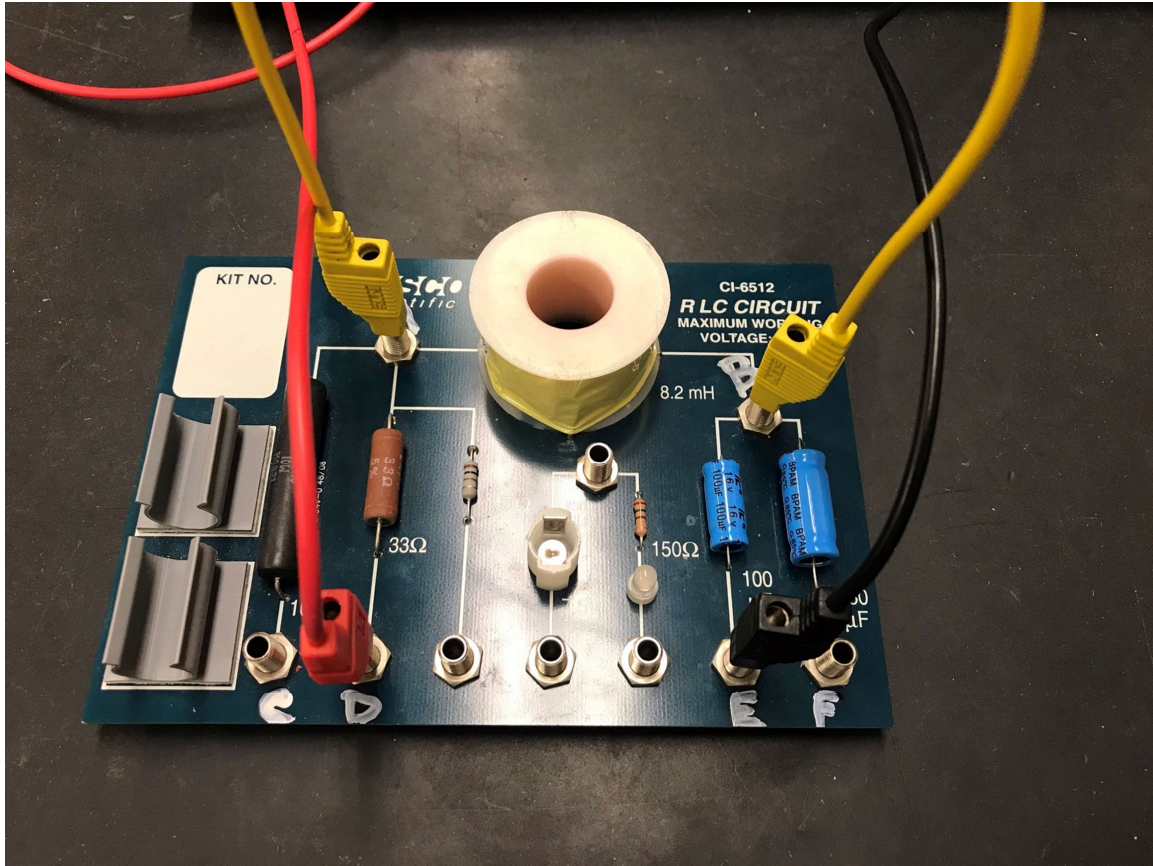


- 2.3. Use equation (10) to describe what would be the expected change in the impedance Z when the frequency f increases. You need to analyze parts of the equation and explain how they contribute to the conclusion, which could be: an *increase of the impedance* or a *decrease of the impedance*.
- 2.4. Read and record the amplitude of current - “Maximum Current” and the amplitude of voltage - “Maximum Voltage” (this should be very close to 4.5 V). Calculate the measured impedance Z using Eq. (8).
- 2.5. Change the frequency f to the next value and repeat measurements and calculations listed in step 2.4.

Activity 3: Impedance as a Function of Frequency for RC Circuits

In this activity, you will add a capacitor to the circuit and measure the impedance of an RC circuit as a function of frequency. Later, you will compare your results with the theoretical value of the impedance calculated using the resistance R and the capacitance C in the RC circuit.

- 3.1. We are going to use $33.0\ \Omega$ resistor again, so move the wire from terminal C back to terminal D on the board. You need to connect $100\ \mu\text{F}$ capacitor to the circuit and eliminate the coil. To do that, connect the two wires from “OUTPUT” terminals of the data interface to terminals **D** and **E** on the RLC board. To bypass (eliminate) the coil from the circuit use an additional wire and plug it into terminals **A** and **B**.



The RC circuit includes now:

- $R_{resistor} = 33.0 \Omega$;
- no coil;
- $C = 100 \mu\text{F}$ capacitor.

Record the resistance R in the RC circuit. $R = R_{resistor} = 33.0 \Omega$ Also, record the value of the capacitance $C = 100 \mu\text{F}$.

- 3.2 Load the software for Activity 3. This software will apply the alternating voltage ($V_{amplitude} = 5.0 \text{ V}$) to the RC circuit. The initial frequency is equal to 10 Hz.
- 3.3. Use equation (13) to describe what would be the expected change in the impedance Z when the frequency f increases. You need to analyze parts of the equation and explain how they contribute to the conclusion, which could be: an *increase of the impedance* or a *decrease of the impedance*.
- 3.4. Read and record the amplitude of current - “Maximum Current” and the amplitude of voltage - “Maximum Voltage” (this should be very close to 5.0 V). Calculate the measured impedance Z using Eq. (8).

- 3.5. Change the frequency f to the next value (20 Hz) and repeat measurements and calculations listed in step 3.4.

Activity 4: Resonance in RLC Circuits

In this activity, you will need both the capacitor C and the inductor L . The goal is to find the resonance frequency, i.e., the frequency that corresponds to the minimum impedance of the single-loop RLC circuit (i.e., the maximum current amplitude). Later, you will compare your results with the theoretical value of the impedance calculated using the inductance L and the capacitance C in the RLC circuit.

- 4.1. This time we are not going to use any external resistors. **Remove** the "bypass" wire connecting terminals **A** and **B**. Move the wire from terminal **D** to terminal **A** on the board. We are going to keep the 100 μF capacitor in the circuit, so keep the wire connected to the terminal **E** on the RLC board. Remember, that even without external resistor, we still have a nonzero resistance in the circuit due to the resistance of the coil R_{coil} .

This time the RLC circuit includes:

- no external resistors;
- a coil with inductance L_{AV} and resistance R_{coil} ;
- $C = 100 \mu\text{F}$ capacitor.

There are no external resistors, so the total resistance R in the RC circuit is equal to the resistance of the coil $R = R_{coil}$. Also, record the value of the capacitance $C = 100 \mu\text{F}$.

- 4.2. Load the software for Activity 4. This software will apply the alternating voltage ($V_{amplitude} = 1.50 \text{ V}$) to the RLC circuit. The initial frequency is equal to 100 Hz.
- 4.3. Read and record the amplitude of current - "Maximum Current" and the amplitude of voltage - "Maximum Voltage" (this should be very close to 1.50 V). Calculate the measured impedance Z using Eq. (8).
- 4.4. Change the frequency f to the next value and repeat measurements and calculations listed in step 4.3.
- 4.5. Prepare a graph of the measured impedance Z versus the frequency f . Find the frequency that corresponds to the minimum impedance, i.e., find the resonance frequency f_{res} . **Print** the graph and attach it to your lab report.
- 4.6. Calculate the theoretical value of the resonance frequency f_0 according to Eq. (17). Calculate the percent difference between the measured resonance frequency f_{res} and the theoretical resonance frequency f_0 .

- 4.7. What should be the theoretical value of the minimum impedance for the frequency equal to the resonance frequency?
- 4.8. Disconnect all wires from the “RLC Circuit” board.

Complete the lab report and return it to the lab TA.

Make sure to complete the following tasks:

- You must submit the answers to the prelaboratory questions online. (3.5 points)
1. One graph from *Activity 1* and one graph from *Activity 4*. (2×1 p. = 2 points)
(Write the title as well as student names and on each graph.)
 2. Your completed Data Sheets. (4.5 points)
 3. Return the completed lab report to your lab TA.